1. *Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(100, \sigma^2)$ where σ^2 is unknown. Consider testing at level α,*

$$H_0 : \sigma^2 \leq 20 \text{ versus } H_1 : \sigma^2 > 20.$$

   (a) *Show that the conditions required for the existence of UMP test are satisfied here.*

   (b) *Derive UMP test of level α.*

   (c) *Consider the test which rejects H_0 whenever $\sum_{i=1}^{n}(X_i - \bar{X})^2 > C$ where $C > 0$ is such that $\sup_{\sigma^2 \leq 20} P(\sum_{i=1}^{n}(X_i - \bar{X})^2 > C) = \alpha$. Show that this test is not UMP test of level α.*

Solution: The joint pdf of $X_1, X_2, \ldots, X_n$ is

$$f(\mathbf{x}|\sigma^2) = \frac{1}{\sqrt{2\pi}^n (\sigma^2)^n} exp\Big(- \frac{\sum_{i=1}^{n}(x_i - 100)^2}{2\sigma^2} \Big).$$

The sufficient statistics for σ^2 is $Y = \sum_{i=1}^{n}(X_i - 100)^2$.

(a) The distribution of $\sum_{i=1}^{n}(X_i - 100)^2/\sigma^2$ has a χ^2 distribution with n degrees of freedom. The distribution function of Y is

$$g(y|\sigma^2) = \frac{1}{2^{\frac{n}{2}} (\sigma^2)^{\frac{n}{2}} \Gamma(\frac{n}{2})} y^{\frac{n}{2}-1} exp(-\frac{y}{2\sigma^2}), y > 0.$$

The family of pdfs $\{g(y|\sigma^2) : \sigma^2 > 0\}$ has a monotone likelihood ratio (MLR), as for every $\sigma_2^2 > \sigma_1^2$, the ratio $g(y|\sigma_2^2)/g(y|\sigma_1^2)$ is a increasing function of y on $\{y : g(y|\sigma_2^2) > 0 \text{ or } g(y|\sigma_1^2) > 0\}$. Using the theorem due to Karlin and Rubin, for any y_0, the test that rejects H_0 if and only if $Y > y_0$ is a UMP level α test, where $\alpha = P_{\sigma^2=20}(Y > y_0)$.

(b) Choosing $y_0 = 20\chi^2(n)(1-\alpha)$ (where $\chi^2(n)(1-\alpha)$ is the $100 \times (1-\alpha)^{th}$ percentile of a χ^2 distribution with $(n-1)$ degrees of freedom), we get $P_{\sigma^2=20}(\sum_{i=1}^{n} Y > y_0) = \alpha$. The UMP test of level α rejects H_0 if and only if $Y > 20\chi^2(n)(1-\alpha)$.

(c) Consider testing the hypothesis $H_0^\star : \sigma^2 = 20$ against $H_1^\star : \sigma^2 = \sigma_1^2$, for some $\sigma_1^2 > 20$. Using the Neyman- Pearson lemma, the MP level α test rejects $H_0^\star$ for $H_1^\star$ if and only if $Y > 20\chi^2(n)(1-\alpha)$. Any other level α test having a power as high as the former must have the same rejection region except for a set A satisfying $\int_A f(\mathbf{x}|\sigma^2)d\mathbf{x} = 0$.
Therefore, the test given in (c) of the question is not a UMP test for testing H_0 against H_1.

$\square$

2. *Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with density $f(x|\lambda) = \lambda exp(-\lambda x), x > 0$, where $\lambda > 0$ is unknown. For testing*

$$H_0 : \lambda = 1 \text{ versus } H_1 : \lambda \neq 1,$$

find the generalized likelihood ratio test at the significance level α.

Solution: The parameter space $\Theta = \{\lambda : \lambda > 0\}$. The parameter space under H_0 is $\Theta_0 = \{\lambda : \lambda = 1\}$.

The likelihood function is

$$L(\lambda|\mathbf{x}) = \lambda^n exp(-\lambda \sum_{i=1}^{n} x_i).$$

The LRT statistic is

$$\phi(\mathbf{x}) = \frac{\underset{\Theta_0}{sup} L(1|\mathbf{x})}{\underset{\Theta}{sup} L(\lambda|\mathbf{x})}.$$

The M.L.E of λ over Θ is $\lambda_n = 1/\bar{x}_n = 1/(\sum_{i=1}^{n} x_i/n)$.

The LRT statistic is

$$\phi(\mathbf{x}) = (\sum_{i=1}^{n} x_i/n)^n exp(-(\sum_{i=1}^{n} x_i - n)).$$

The generalized likelihood ratio test of significance level α that rejects H_0 is given as

$$\begin{aligned} \phi(\mathbf{x}) &= 1, \text{ for } (\textstyle\sum_{i=1}^{n} x_i/n)^n exp(-(\textstyle\sum_{i=1}^{n} x_i - n)) < c_\alpha \\ &= 0 \text{ otherwise,} \end{aligned}$$

where c_α is such that $P_{H_0}((\sum_{i=1}^{n} X_i/n)^n exp(-(\sum_{i=1}^{n} X_i - n)) < c_\alpha) = \alpha$. The above critical region for generalized likelihood ratio test of size α that rejects H_0 can be written as

$$\bar{x}_n < c'_{1,\alpha}, \text{ and } \bar{x}_n > c'_{2,\alpha},$$

where $c'_{1,\alpha}$ and $c'_{2,\alpha}$ are chosen such that

$$P_{H_0}(\sum_{i=1}^{n} X_i < c'_{1,\alpha}, \sum_{i=1}^{n} X_i > c'_{2,\alpha}) = \alpha.$$

$Y = \sum_{i=1}^{n} X_i$ follows a Gamma distribution, with pdf

$$g(y|\lambda) = \frac{\lambda^n}{\Gamma(n)} y^{n-1} exp(-y\lambda), y > 0.$$

For an equal tail test, $c'_{1,\alpha}$ and $c'_{2,\alpha}$, respectively, are the $\frac{\alpha}{2} \times 100^{th}$ and $(1 - \frac{\alpha}{2}) \times 100^{th}$ percentile of $Gamma(n, \lambda)$ distribution with pdf $g(y|\lambda)$.

□

3. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. $N(\mu, \sigma^2)$, where $\mu \geq 0$ and $\sigma^2 > 0$. Let $\theta = (\mu, \sigma^2)$.*

   (a) *What is the parameter space Θ in this model?*

   (b) *Find the m.l.e., $(\hat{\mu}, \hat{\sigma}^2)$ of (μ, σ^2).*

   (c) *Find the UMVUE $\hat{\mu}^\star$ of μ.*

   (d) *Show that $E((\hat{\mu} - \mu)^2) \leq E((\hat{\mu}^\star - \mu)^2)$ for all $\theta \in \Theta$.*

Solution:

(a) The parameter space $\Theta = \{(\mu, \sigma^2) : \mu \geq 0, \sigma^2 > 0\}$.

(b) the log-likelihood for estimating θ can be written as

$$L(\theta|\mathbf{x}) = log(f(\theta|\mathbf{x})) = C - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2} - \frac{n}{2}log(\sigma^2),$$

where C is independent of θ. The partial derivative, with respect to μ is

$$\frac{\partial L(\theta|\mathbf{x})}{\partial \mu} = \sum_{i=1}^{n}\frac{x_i-\mu}{\sigma^2}.$$

Setting the partial derivative to 0 and solving the equation yield the following solution

$$\hat{\mu} = \sum_{i=1}^{n}\frac{x_i}{n} = \bar{x}_n.$$

Next, to obtain the mle of θ.

$$S(\mathbf{x}) = \sum_{i=1}^{n}(x_i-\mu)^2 = \sum_{i=1}^{n}(x_i-\bar{x}_n)^2 + n(\bar{x}_n-\mu)^2$$

When $\bar{x}_n < 0$, $S(\mathbf{x})$ is increasing in μ for $\mu \geq 0$. Therefore, for $\bar{x}_n < 0$, for any value of σ^2, $L(\theta|\mathbf{x})$ is maximized at $\hat{\mu} = 0$. While for $\bar{x}_n \geq 0$, the sum $S(\mathbf{x})$ is minimum at $\hat{\mu} = \bar{x}_n$. The MLE for μ is

$$\hat{\mu} = 0 \text{ for } \bar{X}_n < 0 \text{ and } \hat{\mu} = \bar{X}_n \text{ for } \bar{X}_n \geq 0.$$

From the above, we only need to show that $\frac{1}{(\sigma^2)^{n/2}}exp\Big(-(\sum_{i=1}^{n}(x_i-\hat{\mu})^2)/2\sigma^2\Big)$ attains its maximum at $\frac{1}{n}\big(\sum_{i=1}^{n}(x_i-\hat{\mu})^2\big)$. Let

$$log(g_1(\sigma^2|\mathbf{x})) = -\frac{\sum_{i=1}^{n}(x_i-\hat{\mu})^2}{2\sigma^2} - \frac{n}{2}log(\sigma^2).$$

Then, setting the derivative of this function with respect to σ^2 to 0, yields the unique solution $\frac{1}{n}\big(\sum_{i=1}^{n}(x_i-\hat{\mu})^2\big)$. Also,

$$\frac{d^2 log(g_1(\sigma^2|\mathbf{x}))}{d(\sigma^2)^2}\bigg|_{\sigma^2=\frac{1}{n}\left(\sum_{i=1}^{n}(x_i-\hat{\mu})^2\right)} < 0.$$

Therefore, the MLE of σ^2 is $\frac{1}{n}\big(\sum_{i=1}^{n}(x_i-\hat{\mu})^2\big)$.

(c) The statistic $(\sum_{i=1}^{n}X_i^2, \sum_{i=1}^{n}X_i)$ is complete. To find the UMVUE for μ we only need to look for an unbiased estimator for μ based on the statistic. As $E(\bar{X}_n) = \mu$, UMVUE $\hat{\mu}^\star$ of μ is $\bar{X}_n$.

(d)

$$\begin{aligned}(\hat{\mu}^\star-\mu)^2 &= (\bar{X}_n-\mu)^2 I(\bar{X}_n \geq 0) + (\bar{X}_n-\mu)^2 I(\bar{X}_n < 0)\\ &= (\bar{X}_n-\mu)^2 I(\bar{X}_n \geq 0) + \mu^2 I(\bar{X}_n < 0) + \bar{X}_n(\bar{X}_n-2\mu)I(\bar{X}_n < 0)\\ &= (\hat{\mu}-\mu)^2 + \bar{X}_n(\bar{X}_n-2\mu)I(\bar{X}_n < 0) \geq (\hat{\mu}-\mu)^2,\end{aligned}$$

as $\bar{X}_n I(\bar{X}_n < 0) < 0$ and $(\bar{X}_n-2\mu)I(\bar{X}_n < 0) < 0$ for $\mu \geq 0$. Hence,

$$E(\hat{\mu}^\star-\mu)^2 \geq E(\hat{\mu}-\mu)^2.$$

$\square$

4. *The weekly number of fires X in a town has the Poisson(θ) distribution. The number of fires observed for five weekly periods were $0, 1, 1, 0, 0$. Assume that these observations are independent, and that the prior distribution on θ is $\pi(\theta) \propto \theta exp(-10\theta) I_{(0,\infty)}(\theta)$.*

   (a) *Derive the posterior distribution θ given the data.*

   (b) *Find the posterior mean and posterior standard deviation of θ.*

   Solution: The distribution of X is

$$f(x|\theta) = \frac{exp(-\theta)\theta^x}{x!}, x = 0, 1, 2, \ldots.$$

   The prior distribution of θ is a Gamma distribution given by

$$\pi(\theta) = \frac{10^2}{\Gamma(2)} \theta exp(-10\theta), 0 < \theta < \infty.$$

   The joint distribution of (X, θ) is

$$g(x, \theta) = f(x|\theta) \times \pi(\theta).$$

$$p(x) = \int_0^\infty g(x, \theta) d\theta = \left(\frac{10}{11}\right)^2 (x + 1).$$

   (a) The posterior distribution θ given the data is

$$g_1(\theta|X = 2) = \frac{11^2 \theta^3 exp(-11\theta)}{\Gamma(4)}.$$

   The posterior distribution is Gamma distribution with scale parameter $1/11$ and shape parameter 4.

   (b) The posterior mean is $4/11$ and posterior standard deviation is $2/11$.

$\square$

5. *Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson(λ), $\lambda > 0$, and let $Y_i = 1$ when $X_i > 0$, and 0 otherwise, $i = 1, 2, \ldots, n$.*

   (a) *Show that $\bar{X}$ is a consistent estimator of λ, and it is asymptotically normally distributed.*

   (b) *Find a transform, $g(\bar{Y})$, of $\bar{Y}$ which is a consistent estimator of λ; derive its asymptotic distribution.*

   (c) *Compare the asymptotic relative efficiency of $\bar{X}$ with respect to $g(\bar{Y})$.*

   Solution: The pmf of X_1 is

$$p(x|\lambda) = \frac{\lambda^x exp(-\lambda)}{x!}, x = 0, 1, 2, \ldots.$$

   The log likelihood function is

$$logL(\lambda|\mathbf{x}) = \sum_{i=1}^{n} x_i log(\lambda) - n\lambda - nlog(x!).$$

(a) The partial derivatives, with respect to λ is

$$\frac{\partial L(\lambda|\mathbf{x})}{\partial\lambda} = \sum_{i=1}^{n} \frac{x_i}{\lambda} - n.$$

Setting these partial derivatives to 0 and solving the equations yield the following unique solution

$$\hat{\lambda} = \sum_{i=1}^{n} \frac{x_i}{n} = \bar{x}_n.$$

Also,

$$\frac{d^2 log(L(\lambda|\mathbf{x}))}{d\lambda^2}\Big|_{\lambda=\hat{\lambda}} < 0.$$

The mle of λ is $\bar{X}$.

(b) The probability distribution of X_1 satisfies the regularity conditions. Hence, the estimator $\hat{\lambda}$ is consistent and

$$\sqrt{n}\big(\hat{\lambda} - \lambda\big) \to N(0, \frac{1}{I_X(\lambda)}), \text{ as } n \to \infty,$$

where $I_X(\lambda)$ is the Fisher's information number obtained as

$$I_X(\lambda) = E\Big(\frac{\partial p(x|\lambda)}{\partial\lambda} log(p(x|\lambda))\Big)^2 = \frac{1}{\lambda}.$$

(c) $P(Y_i = 0) = exp(-\lambda)$ and $P(Y_i = 1) = 1 - exp(-\lambda)$. The joint distribution of $Y_1, \ldots, Y_n$ is

$$h(\mathbf{y}|\lambda) = exp(-\lambda \sum_{i=1}^{n} y_i)(1 - exp(-\lambda))^{n-\sum_{i=1}^{n} y_i}$$

The log likelihood function is

$$logL_1(\lambda|\mathbf{y}) = -\lambda \sum_{i=1}^{n} y_i + \frac{(n - \sum_{i=1}^{n} y_i)}{(1 - exp(-\lambda))}.$$

The partial derivative, with respect to λ is

$$\frac{\partial L_1(\lambda|\mathbf{y})}{\partial\lambda} = \sum_{i=1}^{n} x_i + \frac{(n - \sum_{i=1}^{n} y_i)exp(-\lambda)}{(1 - exp(-\lambda))}$$

Setting these partial derivatives to 0 and solving the equations yield the following unique solution

$$\hat{\lambda}_1 = -log(\sum_{i=1}^{n} \frac{y_i}{n}).$$

Also,

$$\frac{d^2 log(L(\lambda|\mathbf{y}))}{d\lambda^2}\Big|_{\lambda=\hat{\lambda}_1} < 0.$$

The mle of λ is $-log(\sum_{i=1}^{n} \frac{Y_i}{n})$, *i.e.* $g(\bar{Y}) = -log(\bar{Y})$.

$$I_Y(\lambda) = \frac{exp(-\lambda)}{(1 - exp(-\lambda))}.$$

The probability distribution of Y_1 satisfies the regularity conditions, and hence

$$\sqrt{n}\big(\hat{\lambda}_1 - \lambda\big) \rightarrow N(0, \frac{1}{I_Y(\lambda)}), \text{ as } n \rightarrow \infty,$$

(c) The A.R.E of of $\bar{X}$ with respect to $g(\bar{Y})$ is

$$A.R.E(\bar{X}, g(\bar{Y})) = \frac{I_Y(\lambda)}{I_X(\lambda)} = \frac{\lambda exp(-\lambda)}{(1 - exp(-\lambda))} = \frac{\lambda}{exp(\lambda) - 1}.$$

This function is strictly decreasing and reaches a maximum value of 1. At $\lambda = 0$, it is 0 and for $\lambda \rightarrow \infty$, the limit of the function is 1.

$\square$